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WAVE LOADS ON SHIPS

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Contents

1 General

1.1 Introduction

1.1.1. The *Informative Publication No. 35/I - Wave loads on Ships* gives a description of the hydrodynamic analysis methods and procedures to support and satisfy requirements given in *Part - II Hull of the Rules for the Classification and Construction of Sea-going Ships* and *Common Structural Rules*.

1.1.2. Prediction of ship motions and loads in waves is one of the most important elements to be considered when designing the ship and anticipating its maritime properties. Due to the high complexity of the physical phenomenon, the prediction is difficult.

The natural method of the prediction is to simulate the behaviour of the ship in irregular wave. However, the need to use very complicated non-linear mathematical models to perform the simulations do not allow to consider the ship behaviour in all possible sea conditions that ship will encounter throughout lifetime. Simulation is possible for a finite, limited amount sea conditions, therefore it is mainly used to carry out the analysis of the non-linear characteristics of the ship behaviour in specific wave conditions. The possibility of considering "all" sea states is provided by linear models and spectral analysis.

When describing any physical phenomenon, simplifying assumptions are made, based on which physical, mathematical and computational models are developed.

It is normally assumed, among others, that:

- sea waves are a stationary ergodic stochastic process, and the Gaussian process, which means that the wave ordinates are subject to a normal distribution with a zero mean value and a variance representing the severity of sea waves;
- the spectral density function is focused around a certain value (the process is narrowbanded);
- the dynamic system ship-wave is a linear system.

These assumptions facilitate the use of spectral analysis to determine the ship motions and ship loads in wave, which comes down to considering the response of ship to the individual harmonics of the wave.

Further basic simplifying assumptions, enabling the use of numerical methods to solve the problem of ship motions in the harmonic component of the wave, relate to hydrodynamic forces. It is assumed that they are linear with respect to the moving surface of the sea and due to the disturbance which the ship introduces to the wave with its presence and its motion. These allows to divide the hydrodynamic forces into:

- Froude-Krylov's forces, caused by undisturbed wave acting on the ship,
- diffraction forces due to the disturbance introduced to waves by a moving ship with a uniform speed,
- forces caused by the ship's oscillating in calm water (radiation forces).

The ship motions and loads in irregular waves are, just like the wave, random. Prediction of the ship motions and loads in waves requires determination of the probabilistic characteristics of these motions and loads.

In general, the rate of change of sea conditions is discretized into a set of sea states. Each of them is assumed to be stationary, has an associated probability of occurrence, and is defined by values of the appropriate parameters such as significant wave height, characteristic wave period, diraction of wave propagation, wave spectrum, etc.

Response of the ship to the given sea state can be computed for different speeds, heading angles in relation to wave propagation and ship loading conditions. Since the sea condition can be considered stationary only for a limited period of time, the results obtained in such a way are called short-term prediction.

Combining the ship response in various sea states with their probability of occurrence, the long-term prediction of ship motion and loads is obtained.

To predict wave loads and ship motions correctly, many aspects are to be considered properly. The prediction is mainly influenced by the following ones:

- the approach to the short-term response prediction,
- long-term prediction scheme,
- probability of occurrence of sea states,
- action of seafarers at sea, etc.

In the case of frequency-domain the transfer function is computed. This function makes it possible to determine the spectral density function of the ship response to waves, which in turn allows to determine the appropriate probability characteristics. The most important is a variance, because it enables to predict the ship's motions and loads in the wave.

1.2 Symbols

Symbols used in the present Publication:

β - heading angle

θ - roll angle

φ - pitch angle

λ - wave length

ρ - density; salt water = 1.025; fresh water = 1.0

ω - wave frequency

ω_e - encounter frequency

g - acceleration of gravity = 9.81

A - wave amplitude

$\mathbf{C}$ - matrix of restoring forces

B - moulded breadth of ship

$\mathbf{N}$ - damping matrix

C_B - block coefficient

D - moulded depth of ship

f - wave frequency

H - wave height

H_s - significant wave height

LCG - longitudinal centre of gravity from aft perpendicular

m - mass of the ship

$\mathbf{M}$ - mass matrix (including added masses)

p - hydrodynamic pressure

$S_{PM}(\omega)$ - wave spectrum,

T - moulded draught

T - wave period

T_z - zero up-crossing period

TCG - transverse centre of gravity from centre line

u - ship forward speed

VCG - vertical centre of gravity above base line

x - longitudinal distance from origin of the coordinate system

y - horizontal distance from origin of the coordinate system

z - vertical distance from origin of the coordinate system

1.3 Coordinate system and definitions

The coordinate system assumed in the section, as well as denotation of motion are shown in figure 1.

Definitions and abbreviations used in the present Publication:

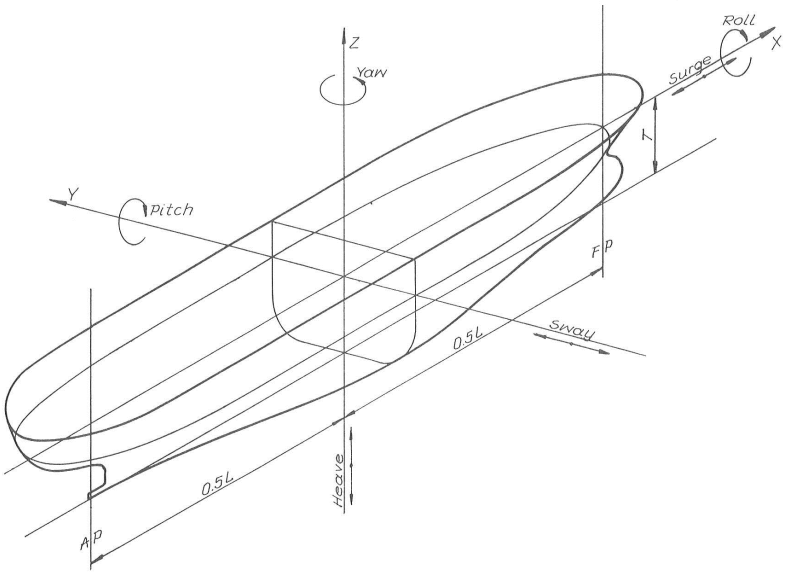

Fig. 1: Definition of coordinate system and denotations of ship motions

D i s p e r s i o n r e l a t i o n - the relationship between the wave period T, in s, and the wave length λ, in m. This relation depends on the water depth d, in m.

R e g u l a r w a v e - two-dimensional wave on sinusoidal water surface, described by the ideal fluid potential flow.

N o n - l i n e a r r e g u l a r w a v e s - asymmetric waves, where the phase velocity depends on wave height.

P e a k p e r i o d T_p - the wave or the period of another response like vertical bending moment, in s, with most energy in the wave or response spectrum.

P h a s e v e l o c i t y c - the propagation velocity of the wave form.

P r o b a b i l i t y d e n s i t y f u n c t i o n - function showing the probability of occurrence at different levels of parameter x, which could be any parameter.

S i g n i f i c a n t w a v e h e i g h t H_s - the average of the highest third wave in a sea state with a duration of 3 hours.

W a v e p e r i o d T - the wave period, in s, is the time interval between successive crests at particular point.

W a v e a m p l i t u d e A - is the waves amplitude below or above the still water surface, in m.

W a v e a n g u l a r f r e q u e n c y - $\omega = {}^{2\pi}/_{T}$, in ${}^{rad}/_{s}$.

W a v e f r e q u e n c y - $f = {}^{1}/_{T}$, in ${}^{1}/_{s}$.

W a v e l e n g t h λ - is the distance between successive crests, in m.

W a v e h e i g h t H - is the distance between the crest and trough within the wave period, in m.

W a v e n u m b e r - $k = {}^{2\pi}/_{\lambda}$, in ${}^{rad}/_{s}$.

Z e r o u p - c r o s s i n g p e r i o d T_z - the wave or response period, in s, between two up-crossings of the zero level in a specific wave/response event or average based on a short term sea state or long term statistics.

1.4 Application

This Informative Publication (Guidance) includes theories and methods for direct calculations of hydrodynamic wave loads and ship motions operating in the North Atlantic environment and its linear and weakly non-linear wave induced responses.

2 Theoretical model for direct analysis of the long-term ship responses to waves

The ship motions and loads in waves and the ship operation are random processes. A mathematical model of the probability of exceeding a given value y_L of the ship response to waves is determined by the following expression:

$$P_r(Y \geq y_L) = \sum_m \sum_l \sum_k p_k p_l p_m \int_{y_l}^{\infty} \int_0^{\infty} \int_0^{\infty} f_{klm}(y|(h_s, t_z)) g(h_s, t_z) dh_s dt_z dy \tag{1}$$

where:

$f_{klm} = f_{klm}(y|(h_s, t_z))$ is the probability density function of the random variable Y (any ship response to waves) in the sea state condition (H_S, T_z), H_S is the significant wave height and T_z is the average zero up crossing wave period, at a certain angle β of ship course in relation to the wave propagation, in a given sea environmental conditions A, and ship loading condition C;

$g = g(h_s, t_z)$ is the probability density function of the sea state occurrence;

p_k is the probability of ship heading γ_k, $k = 1, \ldots, r$, in relation to waves;

p_l is the probability of ship presence in specific sea environmental conditions A_i, $i = 1, \ldots, n$;

p_m is the probability of the ships loading condition occurrence, c_m, $m = 1, \ldots, t$;

y_L is a given number.

Theoretical probability density functions in formula (1), strictly describing the sea states occurrence and ship response to waves, are unknown. Hence, the following assumptions are introduced below.

2.1 Probability density function of short-term ship response to waves

It is assumed that the probability density function of the random variable Y, representing a ship response to waves in a given sea state (short-term response), can be described by the following Rayleigh probability density function:

$$f(y) = \frac{y}{\sigma} \exp\left(-\frac{y^2}{2\sigma}\right). \tag{2}$$

The probability that the amplitude of the response Y will exceed a given value is

$$P(Y > y_L) = \int_{y_l}^{\infty} \frac{y}{\sigma} \exp\left(-\frac{y^2}{2\sigma}\right) dy = \exp\left(-\frac{y_L^2}{2\sigma}\right) \tag{3}$$

where σ is the variance of random process of the ship response to wave.

Theoretically, the Rayleigh distribution represents the probability distribution of amplitudes of random process with narrow-band spectrum. The buoy measurements showed that ocean waves have the following characteristics [4]:

i. waves are considered to be a steady-state ergodic random process;

ii. waves are a Gaussian random process (wave profiles are distributed following the normal probability distribution with zero mean and variance representing the sea severity);

iii. the wave spectral density function is narrow-banded (its spectrum is sharply concentrated at a particular frequency).

Based on the wave characteristics it was proved in [4] that the probability distribution of amplitudes is the Rayleigh distribution. Furthermore, the narrow-banded assumption is a *conservative* approach.

This distribution can also be used to describe ship responses to waves in a given sea state, provided the linear differential equations of motion are used to determine the responses.

After adoption of assumption (i), formula (1) takes the following form:

$$\begin{aligned} P_r(Y \geq y_L) &= \sum_m \sum_l \sum_k p_k p_l p_m \int_0^\infty \int_0^\infty P_{klm}(Y > y_L) g(h_s, t_z) dh_s dt_z \\ &= \sum_m \sum_l \sum_k p_k p_l p_m \int_0^\infty \int_0^\infty \exp\left(-\frac{y_L^2}{2\sigma|(h_s, t_z)} \right)\Bigg|_{klm} g(h_s, t_z) dh_s dt_z \end{aligned} \tag{4}$$

where P_{klm} is the probability (determined by the Rayleigh distribution) that a random ship response amplitude Y will exceed a given value y_L, in a given sea state (H_S, T_z), for the ship heading γ_k in relation to the direction of wave propagation, in a given sea environmental conditions A_i, and in the loading condition C_m.

2.2 Probability of ship presence in specific sea environment

Let the ship sail during its life in a sea of area A which can be divided into separate areas A_i, $i = 1, \ldots, n$:

$$A = \bigcup_{i=1}^{n} A_i.$$

It is assumed that in a suitably short period of time the alternating wave conditions in a given area A_i, $i = 1, \ldots, n$, can be defined by seaway characteriscics as significant wave height H_S and characteristic period T_z.

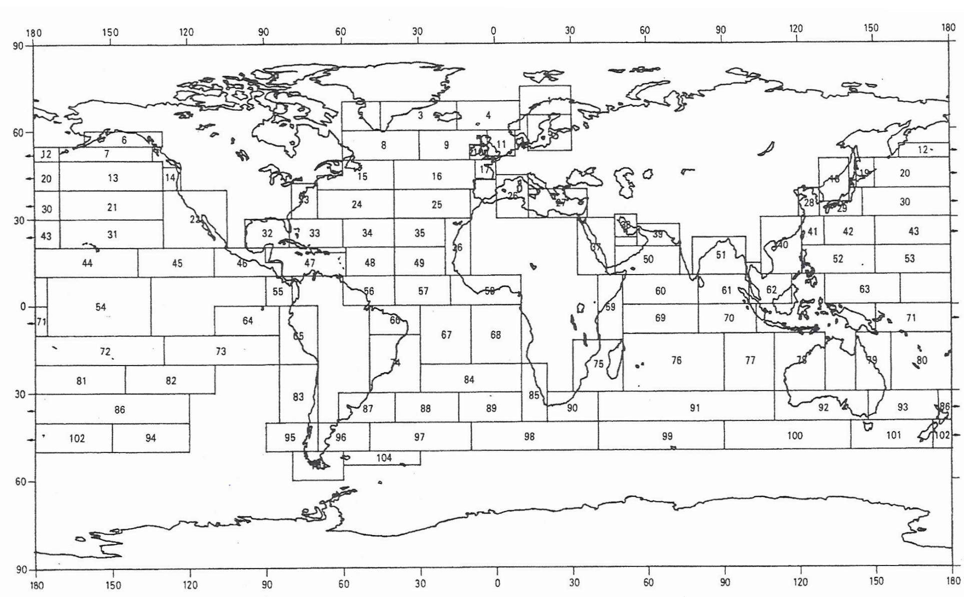

Fig. 2: Nautical zones

According to the Functional Requirement II.2 of IMO GBS [6] the sea environmental conditions A_i, $i = 1, \ldots, s$, is to be the North Atlantic, $i = 1$, covering the zones 8, 9, 15 and 16 of GWS [5], defined by IACS Rec. 34 [1].

The North Atlantic is represented by appropriate probabilities of occurrence of the sea states in this area and by the Pierson-Moskowitz wave spectrum $S_{H_S,T_z}(\omega)$. The Pierson-Moskowitz is one of the simplest models used to describe a fully developed sea state, when a wind is blowing in such a long time that cannot increase the energy in the wave – the energy transfer is balanced by dissipation. This spectrum is a one-parameter spectrum completely described by the wind speed. However, mostly the sea state is not fully developed as the wind speed and direction change, the fetch is too short, or the duration is not long enough, especially for strong winds and high waves. The two-parameter Pierson-Moskowitz spectrum is usually used to develop the irregular wave representing the sea state and is recommended by [1]

$$S_{H_S,T_z}(\omega) = \frac{H_S^2}{4\pi}\left(\frac{2\pi}{T_z}\right)^4 \omega^{-5} \exp\left(-\frac{1}{\pi}\left(\frac{\omega T_z}{2\pi}\right)^{-4}\right) \tag{5}$$

where ω is wave frequency, H_S is significant wave height and T_z is average wave period between zero up-crossings given by:

$$T_z = 2\pi\sqrt{\frac{m_0}{m_2}}$$

where m_0 and m_2 are zero and second spectral moments.

2.3 Probability distribution of the sea state $g(h_s, t_z)$ occurrence

The probabilities p_{ij} of the sea state occurrence given in the form of matrix (scatter diagrams) $[H_{S_i}, T_{z_j}]$, $i = 1, \ldots, n_h$, $j = 1, \ldots, n_t$, are used to approximate the distribution $g = g(h_s, t_z)$ in order to make it possible to compute numerically the long-term probabilities of the ship responses.

H_s/T_z	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5
0.5	0.0	0.0	1.3	133.7	865.6	1186.0	634.2	186.3	36.9
1.5	0.0	0.0	0.0	29.3	986.0	4976.0	7738.0	5569.7	2375.7
2.5	0.0	0.0	0.0	2.2	197.5	2158.8	6230.0	7449.5	4860.4
3.5	0.0	0.0	0.0	0.2	34.9	695.5	3226.5	5675.0	5099.1
4.5	0.0	0.0	0.0	0.0	6.0	196.1	1354.3	3288.5	3857.5
5.5	0.0	0.0	0.0	0.0	1.0	51.0	498.4	1602.9	2372.7
6.5	0.0	0.0	0.0	0.0	0.2	12.6	167.0	69.0	1257.9
7.5	0.0	0.0	0.0	0.0	0.0	3.0	52.1	270.1	594.4
8.5	0.0	0.0	0.0	0.0	0.0	0.7	15.4	97.9	255.9
9.5	0.0	0.0	0.0	0.0	0.0	0.2	4.3	33.2	101.9
10.5	0.0	0.0	0.0	0.0	0.0	0.0	1.2	10.7	37.9
11.5	0.0	0.0	0.0	0.0	0.0	0.0	0.3	3.3	13.3
12.5	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.0	4.4
13.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	1.4
14.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.4
15.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
16.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

H_s/T_z	10.5	11.5	12.5	13.5	14.5	15.5	16.5	17.5	18.5
0.5	5.6	0.7	0.1	0.0	0.0	0.0	0.0	0.0	0.0
1.5	703.5	160.7	30.5	5.1	0.8	0.1	0.0	0.0	0.0
2.5	2066.0	644.5	160.2	33.7	6.3	1.1	0.2	0.0	0.0
3.5	2838.0	1114.1	337.7	84.3	18.2	3.5	0.6	0.1	0.0
4.5	2685.5	1275.2	455.1	130.9	31.9	6.9	1.3	0.2	0.0
5.5	2008.3	1126.0	463.6	150.9	41.0	9.7	2.1	0.4	0.1
6.5	1268.6	825.9	386.8	140.8	42.2	10.9	2.5	0.5	0.1
7.5	703.2	524.9	276.7	111.7	36.7	10.2	2.5	0.6	0.1
8.5	350.6	296.9	174.6	77.6	27.7	8.4	2.2	0.5	0.1
9.5	159.9	152.2	99.2	48.3	18.7	6.1	1.7	0.4	0.1
10.5	67.5	71.7	51.5	27.3	11.4	4.0	1.2	0.3	0.1
11.5	26.6	31.4	24.7	14.2	6.4	2.4	0.7	0.2	0.1
12.5	9.9	12.8	11.0	6.8	3.3	1.3	0.4	0.1	0.0
13.5	3.5	5.0	4.6	3.1	1.6	0.7	0.2	0.1	0.0
14.5	1.2	1.8	1.8	1.3	0.7	0.3	0.1	0.1	0.0
15.5	0.4	0.6	0.7	0.5	0.3	0.1	0.1	0.1	0.0
16.5	0.0	0.2	0.2	0.2	0.1	0.1	0.0	0.1	0.0

Fig. 3: Scatter diagram for North Atlantic operation with H_s and T_z

After adoption the scatter diagram, formula (4) takes the following form [3]:

$$P_r(Y > y_L) = \sum_m \sum_l \sum_k \sum_j \sum_i \exp\left(-\frac{y_L^2}{2\sigma_{ijklm}}\right) p_{ij} p_k p_l p_m. \quad (6)$$

Formula (6) is the numerical model to determine the long-term probability that the amplitude of the ship response will exceed a given value y_L. In the method to determine the long-term ship responses the scatter diagram defined by IACS Rec. 34 [1], covering the zones 8, 9, 15 and 16 of GWS [5] was used.

2.4 Probability distribution of ship heading in relation to the direction of wave propagation

The uniform probability distribution of ship heading γ_k in relation to the direction of wave propagation is recommended by IACS Rec. 34 [1]. However, to better reflect the reality, the 3D distribution which determines the heading distribution for each individual ship was developed, which results were approximated for the following linear function of the ship length [3]:

$$P(L, \gamma) = a(\gamma)(L - 90) + P(90, \gamma), \qquad L[m] \in [90, 350],$$
$$P(L, \gamma) = 0.083333, \qquad L > 350[m],$$
$$a(\gamma) = \frac{0.083333 P(90, \gamma)}{350 - 90},$$

determining the 1D probability distribution for $L > 90[m]$, where $P(90, \gamma)$, $\gamma = 0°, 30°, 60°, \ldots, 33°$ is the ship heading distribution presented in Table 1, approximating 3D distribution for ship of $L = 90[m]$. Vector $[0.083333, 0.083333, \ldots, 0.083333]$ represents the uniform distribution.

Tab. 1: 1D Conditional ship heading distribution $P(90, \gamma)$, $\gamma = 0°, 30°, 60°, \ldots, 33°$ developed for ship of length $L = 90[m]$

γ	0	30	60	90	120	150	180	210	240	270	300	330
p	0.271838	0.040681	0.031733	0.02541	0.051746	0.047857	0.333307	0.047857	0.051746	0.02541	0.031733	0.040681

2.5 Probability of exceeding a ship response to waves

According to the requirement of IMO GBS FR II.1 [6], the specified ship design life is equal to 25 years. This life span determines the number of cycles of the ship response to waves which in turn determines the probability of response exceedance per cycle.

It was assumed in the numerical model determining the long-term ship responses to waves that the probability of exceeding the design value of ship response to waves is 10^{-8}, which approximately corresponds to 25 years of the specified ship design life.

The specified design life according to the IMO GBS FR II.1 [6] (specified also in CSR) is to be 25 years.

The following figure shows the ratio between the 10^{-8} extreme value and the 25 years extreme value for hull girder loads. The ratio $M_{wv}(propb.10^{-8})/M_{wv}(25years)$ is greater than 1 and is increasing with the ship length [3].

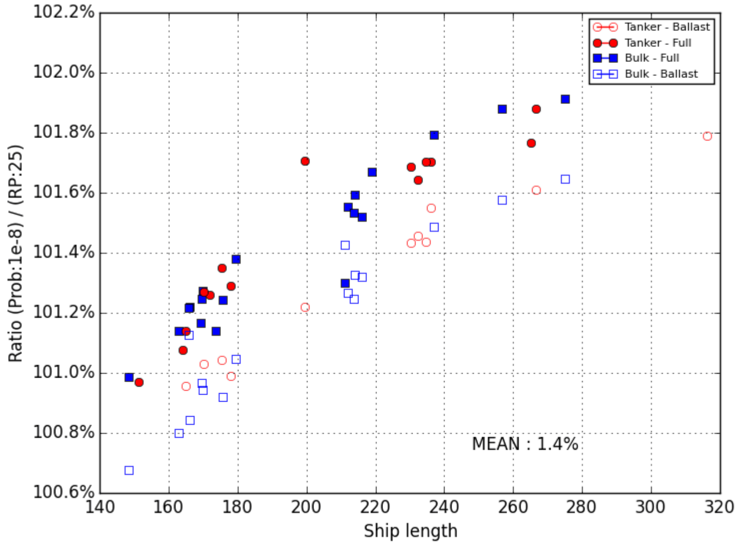

Fig. 4: Ratio $^{M_{wv}(propb.10^{-8})}/_{M_{wv}(25years)}$ at midship

2.6 The ship loading condition

For extreme loading for strength assessment with return period of 25 years, unfavourable loading conditions according to the loading manual should be used.

2.7 Ship forward speed

The ship forward speed v is assumed to be equal to 5 knots for the extreme sea loads design scenario. This speed is the minimum possible speed reducing the dynamic ship response to high waves and assuring the manoeuvrability of the ship.

The maximum expected hull structure response induced by the most severe sea states almost coincides with the long-term predicted value in the range of small exceeding probabilities. In the severe sea states the ship speed is reduced both involuntarily and voluntarily based on good seamanship to the minimum possible speed to assure the manoeuvrability of the ship. As shown in Figure 5 (included in TB Ref. Pt 1, Ch 4, Sec 1 [1.1.5] [8]) which illustrates the speed reduction in severe weather, the speed equal to 5 knots was used to determine the extreme design sea loads scenario.

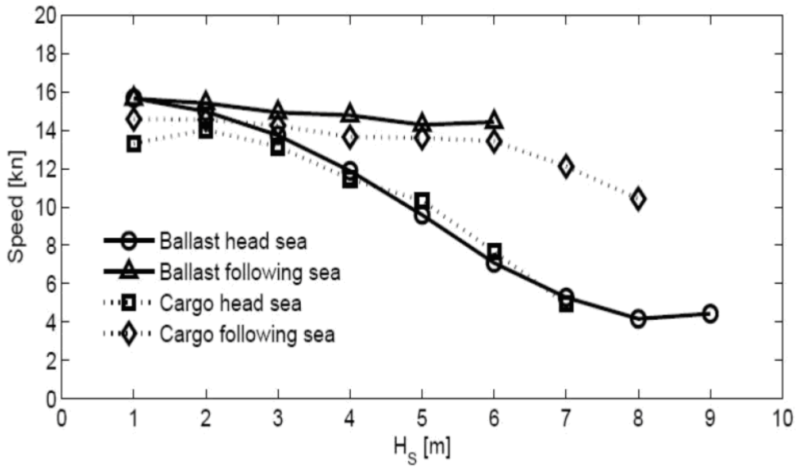

Fig. 5: Speed reduction vs the wave height

3 Spectral analysis

Determination of ship response in the frequency domain.

3.1 Energy spectrum of ship response

The Rayleigh distribution of the random ship response Y to waves is determined using the spectral analysis according to which

- the energy spectrum of ship response $S_Y(\omega)$ is determined by:

$$S_Y(\omega) = Y^2 S_{H_S,T_z}(\omega), \tag{7}$$

where $S_{H_S,T_z}(\omega)$ is wave spectrum for the given sea state determined by H_S and T_z, Y is transfer function;

- variance σ of the considered random process of ship response Y is the zero moment of energy spectrum $S_Y(\omega)$:

$$\sigma = \int_0^\infty \omega^i S_Y(\omega) d\omega, \qquad i = 1, \ldots, n. \tag{8}$$

The variance σ determines the Rayleigh distribution (2) which in turn determines the probability that the amplitude of the ship response Y will exceed a given value y_L (3).

Variance σ of the considered random process of ship response Y is the zero moment of energy spectrum $S_Y(\omega)$ defined for long crested waves. The short crested waves are recommended by [1] for the engineering application, therefore, the energy spectrum $S_Y(\omega, \vartheta)$ and the transfer function Y depend on relative angle ϑ of wave spreading.

In this case the energy spectrum $S_Y(\omega, \vartheta)$ of the considered ship response is determined by:

$$S_Y(\omega, \vartheta) = Y^2 S_{H_S,T_z}(\omega, \vartheta), \tag{9}$$

where $S_{H_S,T_z}(\omega, \vartheta) = S_{H_S,T_z}(\omega) f(\vartheta)$, $S_{H_S,T_z}(\omega)$ is given by (5) and the $f(\vartheta)$given in the following form:

$$f(\vartheta) = \begin{cases} \frac{2}{\pi}\cos^2(\vartheta_0 - \vartheta) & \text{for } \vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

is the angular spreading function, ϑ_0 is the main direction of wave propagation, ϑ is the relative spreading around the main direction of wave propagation. In long term calculations, all wave heading angles $\vartheta \in [-\pi/2, \pi/2]$ can be represented by the headings with increment at most $\Delta\vartheta = \pi/6$ between headings.

The variance σ, which is the zero order of the spectral moments of the ship response for a given heading is described as:

$$\sigma = m_0 = \frac{2}{\pi}\int_0^\infty \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (cos(\vartheta_0 - \vartheta)Y(\vartheta,\omega))^2\right] S_{H_S,T_z}(\omega)d\omega \tag{11}$$

4 Determination of ship response to waves in frequency domain

The non-linear equations of ship motions in irregular wave are described by the following system of differentiational equations:

$$\begin{gathered} m\left(\dot{\mathbf{V}}_Q(t) + \mathbf{\Omega}(t)\times\mathbf{V}_Q(t)\right) = \mathbf{F}(t) \\ \dot{\mathbf{L}} + \mathbf{\Omega}(t)\times\mathbf{L}_Q(t) = \mathbf{M}(t) \\ \dot{\mathbf{R}}_{UQ}(t) = \mathbf{V}_Q - \mathbf{\Omega}(t)\times\mathbf{R}_{UQ}(t) \\ \left(\dot{\varphi}(t),\dot{\theta}(t),\dot{\psi}(t)\right) = D_\Omega^{-1}\mathbf{\Omega}(t), \end{gathered} \tag{12}$$

In order to linearise the forces, on the right side of the equations, so that they can be applied to a linear model, the motion of the ship, the displacement of ship's center of mass, its rotation angles and speed are to be assumed small values. Therefore, the product of them is neglected and after expanding the sine and cosine functions by the Maclarien's series the non-linear elements can also be neglected. Applying such linearisations to the equations (12) the following linear system of equations are obtained:

$$\begin{aligned} m\ddot{x}_{UQ_1} &= \overline{f}_1 \\ m\ddot{x}_{UQ_2} &= \overline{f}_2 \\ m\ddot{x}_{UQ_3} &= \overline{f}_3 \\ J_1\ddot{\varphi} - D_{13}\ddot{\psi} &= \overline{m}_{Q_1} \\ J_2\ddot{\theta} &= \overline{m}_{Q_2} \\ J_3\ddot{\psi} - D_{31}\ddot{\varphi} &= \overline{m}_{Q_3} \end{aligned} \tag{13}$$

Substituting the ship displacements, Froud-Krylov, diffraction, radiation and restoring forces in the following form:

$$\xi_j(t) = \xi_{Aj}e^{i\omega_E t}, \qquad j = 1,\dots,6, \tag{14}$$

to equations of motions (13), the following linear algebraic system of equations, determining the amplitude of the vessel's motion in a regular wave are obtained:

$$\left[-(\omega_E^2 M + C) + i\omega_E N\right]\boldsymbol{\xi}_A = \left[\mathbf{Y}_W + \mathbf{Y}_D\right]_A, \tag{15}$$

where the elements of matrixes M and N have the following form:

$$\begin{aligned} &M_{ii} = m + m_{ii}, \qquad M_{i+3,i+3} = J_i + m_{(i+3)(i+3)}, \qquad i = 1,2,3 \\ &M_{ij} = m_{ij} \qquad \text{for others} \qquad i,j = 1,\dots,6, \\ &N_{ij} = n_{ij}, \qquad j = 1,\dots,6, \end{aligned} \tag{16}$$

m_{ij} are added masses and n_{ij} are damping coefficients. Separating (15) to real and imaginary part, the following set of twelve algebraic equations are obtained:

$$\begin{bmatrix} -(\omega_E^2 M + C) & -\omega_E N \\ \omega_E N & -(\omega_E^2 M + C) \end{bmatrix}\begin{bmatrix} \boldsymbol{\xi}_A^R \\ \boldsymbol{\xi}_A^I \end{bmatrix} = \begin{bmatrix} (\mathbf{Y}_W^R + \mathbf{Y}_D^R)_A \\ (\mathbf{Y}_W^I + \mathbf{Y}_D^I)_A \end{bmatrix}. \tag{17}$$

Solving the above equation, the ship displacement amplitude $\boldsymbol{\xi}_A$ for each ω of exciting wave, we get so-called function $\mathbf{X}$ of ship motions excited by the harmonic wave of amplitude $\zeta_A = 1$ (for angular ship motions the wave slope $k\zeta_A = 1$, $k = {}^{\omega^2}/_g$ is taken into account). Such calculated functions $\mathbf{X}$ is dimensionless and allows to calculate ships motion components using the formula (14):

$$\begin{aligned} \xi_j(t) &= \xi_{Aj}e^{i\omega_E t} = X_j(\omega)\zeta_A e^{i\omega_E t}, \qquad j = 1,2,3, \\ \xi_j(t) &= \xi_{Aj}e^{i\omega_E t} = X_j(\omega)k\zeta_A e^{i\omega_E t}, \qquad j = 4,5,6, \end{aligned} \tag{18}$$

where $\mathbf{X} = (X_1(\omega), \ldots, X_6(\omega))$ is function of excitation, $\omega_E = \omega(1 - \frac{\omega v}{g}\cos\beta)$ is encounter frequency.

4.1 Ship response to regular wave - transfer functions

The ship motion equations (17) were derived on the assumption that the ship motions are excited by regular wave. The transfer function for particular degrees of freedom and their shift phases are determined as follows:

$$\begin{gathered} |X_j(\omega)| = \sqrt{X_j^{R^2}(\omega) + X_j^{I^2}(\omega)}, \quad \delta_j(\omega) = \operatorname{arctg}\frac{X_I(\omega)}{X_R(\omega)}, \\ j = 1,\ldots,6, \quad \omega \in (0,\infty), \end{gathered} \tag{19}$$

The vertical displacement z_w of any point $\overline{x} = (x_1, x_2, 0)$ of ship in relation to the wave surface is given by the formula:

$$z_w = \xi_3 - (x_1 - x_{G_1})\xi_5 + x_2\xi_4 - \zeta, \tag{20}$$

where ξ_3 is vertical displacement of center of ship mass, $\xi_5 = \theta$ and $\xi_4 = \phi$ are Euler angles related to pitch and roll, x_{G1} is the center of ship mass, and $\zeta = \zeta_A e^{-i\mathbf{k}\cdot\overline{x}}e^{i\omega_E t}$ is ordinate of the wave surface at point $(x_1, x_2, 0)$. Substituting functions (18) to equation (20) yields:

$$z_w = \left[X_3 - (x_1 - x_{g1})kX_5 + x_2kX_4 - e^{-i\mathbf{k}\cdot\overline{\mathbf{x}}}\right]\zeta_A e^{i\omega_E t}, \qquad \omega \in (0,\infty). \tag{21}$$

The expression in square brackets is therefore a complex function of the excitation of relative ship motions, which is denoted by Z_w. It is necessary to know the excitation function of heave, pitch and roll motions to determine the Z_w. The modulus of this function is transfer function of relative motions. Differentiation of (21) in respect to time results in the transfer function of relative velocity.

Horizontal longitudinal acceleration of ship a_{HL}, horizontal transversal acceleration a_{HT} and vertical acceleration a_V at any point of the ship, excited by the wave acting on the cargo or ship structure, are given by the formulas:

$$\begin{aligned} a_{HL}(x,t) &= -\left[\ddot{\xi}_1 + (x_3 - x_{G3})\ddot{\xi}_5 - x_2\ddot{\xi}_6\right] \\ a_{HT}(x,t) &= -\left[\ddot{\xi}_2 + (x_1 - x_{G1})\ddot{\xi}_6 + (x_3 - x_{G3})\ddot{\xi}_4\right] \\ a_V(x,t) &= -\left[\ddot{\xi}_3 - (x_1 - x_{G1})\ddot{\xi}_5 - x_2\ddot{\xi}_4\right], \end{aligned} \tag{22}$$

where $\ddot{\xi}_1$, $\ddot{\xi}_2$ i $\ddot{\xi}_3$ are accelerations of center of ship mass, and $\ddot{\xi}_4 = \ddot{\Phi}$, $\ddot{\xi}_5 = \ddot{\theta}$ and $\ddot{\xi}_6 = \ddot{\psi}$ are angular accelerations.

From equation (18) follows that $\ddot{\xi}_j(t) = -\omega_E^2\xi_j(t)$, $j = 1,\ldots,6$. Substituting this into formula (22), yields:

$$\begin{aligned} a_{HL}(x,t) &= a_{HL}(x)\zeta_A e^{i\omega_E t}, \\ a_{HT}(x,t) &= a_{HT}(x)\zeta_A e^{i\omega_E t}, \\ a_V(x,t) &= a_V(x)\zeta_A e^{i\omega_E t}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} a_{HL}(x) &= \omega_E^2\left[X_1 + (z - z_g)kX_5 - ykX_6\right], \\ a_{HT}(x) &= \omega_E^2\left[X_2 + (x - x_g)kX_6 + (z - z_g)kX_4\right], \\ a_V(x) &= \omega_E^2\left[X_3 - (x - x_g)kX_5 - ykX_4\right], \end{aligned} \tag{24}$$

and similarly as in the case of relative displacement, $a_{HL}(x)$, $a_{HT}(x)$ and a_V are functions of accelerations excitation and its modulus are transfer functions.

4.2 Pressure transfer functions

Transfer functions of dynamic pressures acting on a specific point of the ship's wetted surface are determined by the formula

$$p - p_a = -\rho \left(\frac{\partial \phi}{\partial t} - u_0 \frac{\partial \phi}{\partial x_1} \right) - g x_3.$$

The dynamic part of the pressure is determined by:

$$p(x,t) = p(x) \zeta_A e^{i \omega_E t},$$

where

$$p(x) = \rho g [e^{k x_3 - i \mathbf{k} \cdot \overline{x}} + i \frac{\omega}{g} \varphi_D - (X_3 + x_2 X_2 - x_1 k X_5) + \nu \sum_{j=1}^{6} X_j \varphi_{Rj}], \qquad x \in S_0, \tag{25}$$

and φ_{Rj}, $j = 1, \ldots, 6$, φ_D are directly calculated from hydrodynamic theory. The modulus of $p(x)$ function is pressure transfer function.

5 Equation of ship motions in waves

5.1 Introduction

From the physical point of view force **f** and moment **m** determine the considered system of ship motion equations. For a ship moving on waves, these functions mainly represent hydrodynamic forces generated by the waves and the ship motions.

Ship motions in waves is a very complex phenomenon and its analysis can be made if simplifying assumptions are introduced. These simplifications apply to the wave, to the flow around the ship moving in the waves and to the ship motion. Depending on the type of simplifications two models are developed:

- simulation of ship motions in irregular waves,
- performing a spectral analysis of ship motion in waves using a dynamic ship-wave system.

5.2 Basic assumptions

Solution of the general differential problem (Navier-Stokes equation) describing physical phenomenon is very difficult. Therefore, to develop theoretical model that allows determination of hydrodynamic forces acting on the ship, the following assumptions and simplifications are introduced:

- the ship moves whit a steady speed in an unlimited depth of water,
- water is incompressible,
- water is ideal fluid,
- the flow around the ship's hull is irrotational.

The flow that meets the above assumptions is a potential flow. The assumptions introduced result in mathematical problem allowing to determine the flow in two steps:

- in the first one, the velocity potential of the water ϕ is determined as the solution of the hydrodynamic boundary problem,
- in the second one, the pressure is determined from linear part of Bernulli equation:

$$p - p_a = -\rho \left(\frac{\partial \phi}{\partial t} - u_0 \frac{\partial \phi}{\partial z_1} \right) - \rho g x_3, \tag{26}$$

where u_0 is mean value of ship's speed and $x = (x_1, x_2, x_3)$.

In mathematical model describing the simulation of ship motion in irregular waves further simplifications are adopted. It was assumed that the following forces act on the ship:

- Froude - Krilov forces generated by undisturbed wave;
- diffraction forces; disturbance that the ship brings to the water;
- radiation forces; the disturbance caused by ocsillating ship.

It means that the forces generated by each velocity potential can be determined independently and that their superposition can be made (linear model of forces is assumed):

$$\mathbf{f} = \mathbf{f}\left[\phi_W(x,t) + \phi_D(x,t) + \phi_R(x,t)\right] = \mathbf{f}[\phi_W(x,t)] + \mathbf{f}[\phi_D(x,t)] + \mathbf{f}[\phi_R(x,t)] \tag{27}$$

where $\phi_W(x,t)$ is velocity field potential in udisturbed wave, $\phi_D(x,t)$ is diffraction potential field and $\phi_R(x,t)$ is radiation potential field.

Complex velocity potential of harmonic wave is given by formula:

$$\phi_W(x,t) = i\zeta_A \frac{g}{\omega} e^{kx_3 - i\mathbf{k}\cdot\overline{x} + i\omega_E t} = \varphi_W(x)e^{i\omega_E t}, \tag{28}$$

where $x = (x_1, x_2, x_3)$, $\overline{x} = (x_1, x_2, 0)$, $\mathbf{k} = (k_1, k_2, 0) = (k\cos\beta, k\sin\beta, 0)$, $k = \omega^2/g$ is wave number, $\omega_E = \omega\left(1 - \frac{u_0\omega}{g}\cos\beta\right)$ is encounter frequency and β is angle between wave vector $\mathbf{k}$ and ship speed vector $\mathbf{u}$.

However, to determine the disturbance that the ship brings to the wave, further simplifying assumptions are introduced to the mathematical model:

- the disturbance that the ship brings into the wave can be approximated by the disturbance which is caused by non-oscillating ship (diffraction),
- the disturbance which is caused by ship motion in waves can be approximated by disturbance which is caused by oscillating ship on calm water (radiation),
- the diffraction velocity field can be described by the following formula

  $$\phi_D(x,t) = \varphi_D(x)e^{i\omega_E t} \tag{29}$$

  where φ_D is complex function of real variable.

From the above analysis it follows that the hydrodynamic forces depend on the ship motions. In order to linearize such forces so that they can be applied to the linear model, the ship motions must also be linear.

The linear equations of ship motions is used in the spectral analysis to determine the transfer function, which determines the vessel motion amplitude in the result of the action of regular wave with a unit amplitude. The ship motion caused by a wave of unit amplitude is small in relation to the dimensions of the ship, therefore it is possible to perform the linearization of radiation forces. In this case addition assumptions are imposed on the radiation velocity field.

The radiation velocity field depends on the velocity of any point p of the wetted surface S of the ship in the normal direction to this surface

$$\begin{aligned} v_{pn} &= (\mathbf{u} + \mathbf{v_p})\cdot\mathbf{n} = (\mathbf{u} + \mathbf{v}_q(t) + \omega(t)\times\mathbf{r}_{qp})\cdot\mathbf{n} \\ &= \mathbf{u}\cdot\mathbf{n} + {}_p(t)\cdot\mathbf{n} + \omega(t)\cdot(\mathbf{r}_{qp}\times\mathbf{n}) \\ &= u_0 n + \sum_{j=1}^{6}\dot{\xi}_j(t)n_j, \end{aligned} \tag{30}$$

where n_j, $j = 1, 2, 3$ are the coordinates of the normal vector, outward to surface S and

$$n_4 = x_2n_3 - x_3n_2, \qquad n_5 = x_3n_1 - x_1n_3, \qquad n_6 = x_1n_2 - x_2n_1,$$

while $\dot{\xi}_j$, $j = 1, \ldots, 6$, are determined by:

$$\dot{\xi}_j = v_{qj}, \qquad j = 1, 2, 3,$$
$$\dot{\xi}_4 \cong \dot{\varphi}, \quad \dot{\xi}_5 \cong \dot{\theta}, \quad \dot{\xi}_6 \cong \dot{\psi}.$$

The v_{pn} velocity described by (30) enables to write the radiation field in the following form:

$$\phi_R(x, t) = \sum_{j=1}^{6} \dot{\xi}_j(t)\varphi_{R_j}(x) \tag{31}$$

where ξ_j, $j = 1, \ldots, 6$, corresponds to particular degree of ship motion.

Based on the form of the excitation force, the ship motion caused by the harmonic wave is harmonic:

$$\xi_j(t) = \xi_{A_j} e^{i\omega_E t} \tag{32}$$

where $\xi_{A_j} = \xi_{A_j}^R + \xi_{A_j}^I$ and ω_E is encounter frequency.

Substituting (31) and (32) to (27) we get dependence for the linear model:

$$\begin{aligned} \mathbf{f} &= \mathbf{f}\left[\varphi_W(x)e^{i\omega_E t} + \varphi_D(x)e^{i\omega_E t} + \sum_{j=1}^{6} \dot{\xi}_j(t)\varphi_{R_j}(x)\right] \\ &= e^{i\omega_E t}\left(\mathbf{f}[\varphi_W(x)] + \mathbf{f}[\varphi_D(x)] + i\omega_E \sum_{j=1}^{6} \xi_{A_j}\mathbf{f}[\varphi_{R_j}(x)]\right). \end{aligned} \tag{33}$$

5.3 Hydrodynamic forces

The generalized hydrodynamic forces that are the result of wave impact on the moving ship are determined by the following equations:

$$\begin{aligned} \mathbf{f}(t) &= -\int_S (p(x, t) - p_a)\mathbf{n}(x, t)dS \\ \mathbf{m}_q(t) &= -\int_S (p(x, t) - p_a)(\mathbf{r}_q(t) \times \mathbf{n}(x, t))dS \end{aligned} \tag{34}$$

where p is pressure at the wetted surface of ship in its momentary position, p_a is atmospheric pressure and $\mathbf{n}$ is normal vector outward to S. The pressure $p - p_a$ is given by (26).

5.4 Linearization of hydrodynamic forces

In the case of small motions of the ship in relation to its equilibrium position higher order terms are neglected. This can be achieved by developing functions that represent the forces or potentials of water velocity according to Taylor's formula. This developments takes the form:

- for scalar function $\phi = \phi(x, t)$:

$$\begin{aligned} \phi(x, t) &= \phi(x_0, t_0) + (\Delta\mathbf{r}_{xx_0} \cdot \nabla\phi)(x_0, t_0) \\ &+ \Delta t \frac{\partial \phi}{\partial t}(x_0, t_0) + O(\Delta r_{xx_0}), \end{aligned} \tag{35}$$

- for vector function $\mathbf{f} = \mathbf{f}(x, t)$:

$$\begin{aligned} \mathbf{f}(x, t) &= \mathbf{f}(x_0, t_0) + [(\Delta\mathbf{r}_{xx_0} \cdot \nabla)\mathbf{f}](x_0, t_0) \\ &+ \Delta t \frac{\partial \mathbf{f}}{\partial t}(x_0, t_0) + O(\Delta r_{xx_0}), \end{aligned} \tag{36}$$

where $\Delta\mathbf{r}_{xx_0}$ is position change of any ship point relative to its average position. With the help of this formula we can represent the integrand functions (34) relative to the wetted surface in the average position of the ship.

Substituting (35), (36), (26) to (34) and neglecting square elements - the following linearized hydrodynamic forces in general form are obtained:

$$\begin{aligned} Y_j \approx y_j &= \int_{S_0} \left(\frac{\partial}{\partial t} - u_0 \frac{\partial}{\partial x_1} \right) \varphi e^{i\omega_E t} n_j ds \\ &= \int_{S_0} \left(i\omega_E - u_0 \frac{\partial}{\partial x_1} \right) \varphi e^{i\omega_E t} n_j ds \end{aligned} \tag{37}$$

where n_j, $j = 1, 2, 3$ are the components of the normal vector outer to surface S and

$$n_4 = x_2 n_3 - x_3 n_2, \qquad n_5 = x_3 n_1 - x_1 n_3, \qquad n_6 = x_1 n_2 - x_2 n_1.$$

Substituting velocity potentials φ presented above to the formula (37) and omitting factor $e^{i\omega_E t}$, the following types of forces which are presented below are obtained.

5.4.1. Froud-Krylov forces

Substituting (28) to (37) the following force components, corresponding to harmonic wave component, are obtained:

$$\begin{aligned} Y_{W_j}^R \approx y_{W_j}^R &= -\rho g \zeta_A \int_{S_0} e^{kx_3} \cos(\mathbf{k} \cdot \overline{x}) n_j ds \\ Y_{W_j}^I \approx y_{W_j}^I &= \rho g \zeta_A \int_{S_0} e^{kx_3} \sin(\mathbf{k} \cdot \overline{x}) n_j ds \end{aligned} \qquad j = 1, \ldots, 6 \tag{38}$$

where the upper indices R and I represent the real and imaginary part of the force.

5.4.2. Diffraction force

Similary to above:

$$\begin{aligned} Y_{D_j}^R \approx y_{D_j}^R &= -\rho \int_{S_0} \left(\omega_E \varphi_D^I + u_0 \frac{\partial}{\partial x_1} \varphi_D^R \right) n_j ds \\ Y_{D_j}^I \approx y_{D_j}^I &= \rho \int_{S_0} \left(\omega_E \varphi_D^R - u_0 \frac{\partial}{\partial x_1} \varphi_D^I \right) n_j ds. \end{aligned} \qquad j = 1, \ldots, 6 \tag{39}$$

5.4.3. Radiation force

Substituting the radiation potential in the form of (35) and (36) into equation (37) yields:

$$Y_{R_j} \approx y_{R_j} = \rho \int_{S_0} \left(i\omega_E - u_0 \frac{\partial}{\partial x_1} \right) \left(\sum_{k=1}^{6} i\omega_E \xi_k \varphi_{R_K} \right) n_j ds. \tag{40}$$

After appropriate transformations, this equation takes the form

$$\begin{aligned} Y_{R_j} \approx y_{R_j} = -i\rho\omega_E e^{i\omega_E t} \sum_{k=1}^{6} \xi_{A_k} \int_{S_0} \Bigg(&\left(\omega_E \varphi_{R_k}^I + u_0 \frac{\partial}{\partial x_1} \varphi_{R_k}^R \right) \\ &- i \left(\omega_E \varphi_{R_k}^R - u_0 \frac{\partial}{\partial x_1} \varphi_{R_k}^I \right) \Bigg) n_j ds \qquad j = 1, \ldots, 6. \end{aligned} \tag{41}$$

In order to approximate a linear system of equations describing the ship motion in waves by a system describing the harmonic oscillator, it is assumed that the radiation forces are proportional to the accelerations and speed of the oscillating ship:

$$\begin{aligned} Y_{R_j} \approx y_{R_j} &= -\sum_{k=1}^{6} \left(m_{jk} \ddot{\xi} + n_{jk} \dot{\xi} \right) \\ &= e^{i\omega_E t} \sum_{k=1}^{6} \xi_{A_k} (\omega_E^2 m_{jk} - i\omega_E n_{jk}), \end{aligned} \tag{42}$$

where $m_{jk} = m_{jk}(\omega_E)$ are added masses and $n_{jk} = n_{jk}(\omega_E)$ are damping coefficientts.

Comparison of the right sight of equations (41) and (42) yields:

$$\begin{aligned} m_{jk} &= -\frac{\rho}{\omega_E}\int_{S_0}\left(\omega_E\varphi^R_{R_k} - u_0\frac{\partial}{\partial x_1}\varphi^I_{R_k}\right)n_j ds \\ n_{jk} &= \rho\int_{S_0}\left(\omega_E\varphi^I_{R_k} + u_0\frac{\partial}{\partial x_1}\varphi^R_{R_k}\right)n_j ds. \end{aligned} \tag{43}$$

5.4.4. Restoring force The restoring forces are the result of the difference in force acting on the displaced ship in still water and the weight of the ship. They are determined by 6×6 matrix with elements:

$$\begin{aligned} c_{33} &= -\rho g|S_w|, \\ c_{35} &= c_{53} = \rho g I_{S1}, \\ c_{44} &= -\rho g(M_{V3} + J_{S2}), \\ c_{55} &= -\rho g(M_{V3} + J_{S1}), \end{aligned} \tag{44}$$

where $|S_w|$ is flotation field, I_{S1} is static moment of flotation, J_{S1} and J_{S1} are interia moments of flotation and M_{V3} is static moment of the under water volume of ship.

The remaining elements of the matrix are equal to zero.

5.4.5. Non-linear equations of ship motions Substituting generalized Froude-Krylov, diffraction and radiation forces to (12) and adding gravity force, damping force and rudder force, the following non-linear equations of ship motions are obtained:

$$\begin{aligned} m\left(\dot{\mathbf{V}}_Q(t) + \mathbf{\Omega}(t)\times\mathbf{V}_Q(t)\right) &= \mathbf{F}_W(t) + \mathbf{F}_D(t) + \mathbf{F}^1_R(t) + \mathbf{F}_T(t) \\ \dot{\mathbf{L}} + \mathbf{\Omega}(t)\times\mathbf{L}_Q(t) &= \mathbf{M}_{QW}(t) + \mathbf{M}_{QD}(t) + \mathbf{F}^2_R(t) + \mathbf{M}_T(t) \\ \dot{\mathbf{R}}_{UQ}(t) &= \mathbf{V}_Q - \mathbf{\Omega}(t)\times\mathbf{R}_{UQ}(t) \\ \left(\dot{\varphi}(t), \dot{\theta}(t), \dot{\psi}(t)\right) &= D^{-1}_{\Omega}\mathbf{\Omega}(t), \end{aligned} \tag{45}$$

where $\mathbf{G} = g(\sin\theta, -\cos\theta\sin\varphi, -\cos\theta\cos\varphi)$, g is gravity acceleration, $\mathbf{F}_T(t)$ and $\mathbf{M}_T(t)$ represent damping force and moment with rudder reaction moment.

The solution of the above system, at the given initial conditions, uniquely describes the ship's motion in waves. The solution of these equatins is only possible using numerical methods.

5.4.6. Linear equations of motions The linear equations of motions are presented in section 4, formula (17).

6 Diffraction and radiation problems

In the case of a diffraction or radiation problem, we are dealing with disturbance that ship brings into the waves or oscillating ship brings into undisturbed water. The shape of the underwater part of the ship is also important and the boundary condition on the wetted ship surface must be added to boundary problem. This condition requires, in case of radiation problem, the component v_n of the water velocity, normal to the wetted surface S_0 of the ship, is to be equal to the normal component of velocity v_{sn} of the considered surface point S_0, which can be described by:

$$v_n(x) = v_{sn}(x) \quad \text{where} \quad v_n(x) = \nabla\phi\cdot\mathbf{n} = \frac{\partial\phi}{\partial\mathbf{n}}, \tag{46}$$

where **n** is a unit, normal vector external to the wetted surface of ship S_0. In case of diffraction problem, the normal component of the velocity of the wave reflected from the surface S_0 has an opposite value to the component of the normal velocity of the wave particle. Therefore, the boundary problem describing the diffraction and radiation potential velocity field includes:

1. the Laplace's equation

$$\Delta\varphi(x) = 0, \tag{47}$$

2. the boundary condition

   2.1 on the free surface

$$-\nu\varphi(x) - 2i\tau\frac{\partial\varphi(x)}{\partial x_1} + \frac{1}{k_0}\frac{\partial^2\varphi(x)}{\partial x_1^2} + \frac{\partial\varphi(x)}{\partial x_3} = 0, \tag{48}$$

   which can be denoted:

$$L_{SF}\varphi(x) \equiv \left(-\nu - 2i\tau\frac{\partial}{\partial x_1} + \frac{1}{k_0}\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_3}\right)\varphi(x) = 0, \tag{49}$$

   where $\nu = {}^{\omega_E^2}/_{g}$, $\tau = {}^{\omega_E u_0}/_{g}$ and $k_0 = {}^{g}/_{u_0^2}$;

   2.2 on the ship wetted surface

   - for the diffraction potential

$$\frac{\partial\varphi_D(x)}{\partial\mathbf{n}} = -\frac{\partial\varphi_{W_1}(x)}{\mathbf{n}}, \tag{50}$$

   where $\varphi_{W_1}(x)$ is undisturbed wave potential;

   - for the radiation potential

$$\begin{aligned}\frac{\partial\varphi^R_{R_i}(x)}{\partial\mathbf{n}} &= n_i,\\ \frac{\partial\varphi^I_{R_i}(x)}{\partial\mathbf{n}} &= 0,\end{aligned} \qquad i = 1,\ldots,6, \tag{51}$$

   where n_j, $j = 1,2,3$ are the coordinates of the outer of normal vector to surface S and

$$n_4 = x_2n_3 - x_3n_2, \qquad n_5 = x_3n_1 - x_1n_3, \qquad n_6 = x_1n_2 - x_2n_1.$$

3. the condition in infinity

   3.1 for the ships velocity $u_0 = 0$

   - the radiation condition

$$\lim_{\rho\to\infty}\sqrt{\rho}\left(\frac{\partial\varphi}{\partial\rho} + i\nu\varphi\right) = 0, \tag{52}$$

   - the condition in infinity on free surface

$$\lim_{\rho\to\infty}\sqrt{\rho}|\varphi| \le c, \tag{53}$$

   where $\rho = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, and c is constant;

   3.2 for the ships velocity $u_0 > 0$

   - the condition imposed on the system of waves generated by the source, in the plane $x_3 = 0$
   - the condition in infinity

$$\lim_{\rho\to\infty}\sqrt{\rho}|\varphi| \le c, \tag{54}$$

   where $\rho = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, and c is constant;

4. the condition at the bottom of sea

$$\lim_{x_3 \to -\infty} \frac{\partial \varphi(x)}{\partial x_3} = 0. \tag{55}$$

It is assumed that the solution of this problem is in the form of a single layer potential:

$$\phi(x) = \int_{S_0} \mu(y) E(x,y) ds_y - \frac{u_0^2}{g} \int_l \mu(y) E(x,y) n_1^2 dl_y, \tag{56}$$

where μ is complex density function of source distribution, $l = \partial S_0$ is boundary of wetted surface S_0, and $\mathbf{n} = (n_1, n_2, n_3)$ is normal vector, outward to surface S_0. Function $E(x,y)$ is fundamental solution for this problem, satisfying the Laplace's equation and boundary conditions, except the condition on the wetted surface.

In the physical interpretation, the fundamental solution represents the velocity potential created by a pulsating source with a unit strength, moving with a constant velocity under the free surface of the water. In special cases it will be either a pulsating source or only a source moving with constant speed. The function E which occure in (56) has the following form:

$$E(x,y) = \frac{1}{4\pi} \left[\frac{1}{|x-y|} - \frac{1}{|x-z|} + G(x,z) \right], \tag{57}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (y_1, y_2, -y_3)$,

$$\begin{aligned} |x-y| &= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}, \\ |x-z| &= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3+y_3)^2}, \end{aligned} \tag{58}$$

and G is a harmonic function, which must be determined form Laplace's equation and the conditions in the free surface equation. Components of equation (57) were chosen based on the following theories:

- analysis of free surface condition,
- on the condition that the radiation waves and diffraction effect should disappear at infinity on free surface,
- function $\frac{1}{4\pi}\left[\frac{1}{|x-y|} - \frac{1}{|x-z|}\right]$ is Green function for Dirichlet half space problem.

Adoption of solution in the form of single potential layer reduces the solutions of (47)-(55) to solution of second type Fredholm integral equation of II kind:

$$\frac{1}{2}\mu(x) + \int_{S_0} \mu(y) \frac{\partial}{\partial \mathbf{n}_y} E(x,y) ds_y - \frac{u_0^2}{g} \int_l \mu(y) \frac{\partial}{\partial \mathbf{n}_y} E(x,y) n_1^2(y) dl_y = \vartheta_n(x). \tag{59}$$

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